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On the effective lagrangian in spinor electrodynamics with added violation of Lorentz and CPT-symmetries

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Abstract. We consider quantum electrodynamics with additional coupling of spinor fields to the space-time independent axial vector violating both Lorentz and CPT-symmetries. The Fock–Schwinger proper-time method is used to calculate the one-loop effective action up to the second order in the axial vector and to all orders in the space-time independent electromagnetic field strength. We find that the Chern–Simons term is not radiatively induced and that the effective action is CPT-invariant in the given approximation.

1 Introduction

Although conservation of the Lorentz and CPT-symmetries belongs to the fundamental laws of nature, various extensions of quantum electrodynamics and, more generally, the standard model, with tiny violation of these symmetries have generated current interest in the last decade [1–5]. In the gauge vector sector of quantum electrodynamics a plausible extension is achieved [1] by adding a Chern–Simons term [6] to the conventional Maxwell term in the lagrangian

$$\mathcal{L}(A,k) = \frac{1}{4e^2}\mathrm{Sp}F^2 - k\tilde{F}A, \qquad (1.1)$$

where A^μ is the electromagnetic potential, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor, $\tilde{F}^{\mu\nu} = (1/2)\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ is its dual, Sp is the trace over the Lorentz indices: $\mathrm{Sp}MN = M_{\mu\nu}N^{\nu\mu}$, and the metric of the Minkowski space is chosen as $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. The second term in the right hand side of (1.1) is obviously non-gauge invariant, however, if vector k^μ is space-time independent, then the integral of the second term over the whole space-time is gauge invariant. Consequently, the action and equations of motion of the theory with lagrangian (1.1) are gauge invariant. The position independent vector k^μ selects a preferred direction in space-time, thus violating both the Lorentz and CPT-symmetries. The observation of distant galaxies puts a stringent bound on the value of k^μ: it should effectively vanish [4,7]. An obvious extension of the spinor sector of quantum electrodynamics is

$$\mathcal{L}(\psi,\bar{\psi},A,b) = \overline{\psi}(\mathrm{i}\hat{\partial} - \hat{A} + \hat{b}\gamma^5 - m)\psi, \qquad (1.2)$$

where $\hat{\partial} = \gamma^\mu\partial_\mu$, $[\gamma^\mu,\gamma^\nu]_+ = 2g^{\mu\nu}$ and $\gamma^5 = -\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$. If the vector b^μ is space-time independent, then the natural question arises, whether a Chern–Simons term can be radiatively induced as a result of the interaction of quantized spinor fields in the theory with the lagrangian (1.2). Different answers to this question have been obtained, which can be summarized as follows. Perturbative calculations (and even non-perturbative ones that however are based on the Feynman diagram technique) yield an induced Chern–Simons term with fixed coefficient, but the value of this coefficient differs depending on a concrete calculation scheme [8–12]. This discrepancy is analyzed further, and the claim is made that the Chern–Simons term is induced with finite but intrinsically ambiguous coefficient [13–15]. There are also non-perturbative (non-diagrammatic or functional) approaches which yield either fixed or ambiguous values of the coefficient before the induced Chern–Simons term [16–19]. Finally, there are rather diverse arguments that the Chern–Simons term is not radiatively induced [2,3,20,21].

To shed more light on this problem, we shall compute the effective action of the theory with lagrangian (1.2) in the approximation keeping all orders of the position independent electromagnetic field strength $F_{\mu\nu}$ and up to the second order in the position independent vector b^μ. We use the Fock–Schwinger [22,23] proper-time method and find that, indeed, the Chern–Simons term is not induced. Moreover, the terms linear in b^μ are not present at all, and the effective action is parity invariant.

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2 Effective action and its regularization

The effective action is obtained by integrating out the fermionic degrees of freedom in the theory with lagrangian (1.2):

$$\begin{aligned} \Gamma(A,b) &= -\mathrm{i}\ln\int \mathrm{d}\bar{\psi}\mathrm{d}\psi\exp\left[\mathrm{i}\int \mathrm{d}^4x\mathcal{L}(\psi,\bar{\psi},A,b)\right] \\ &= -\mathrm{i}\ln\mathrm{Det}(\mathrm{i}\hat{\partial}-\hat{A}+\hat{b}\gamma^5-m) \\ &= \int \mathrm{d}^4x\mathcal{L}^{\mathrm{eff}}(A,b), \end{aligned} \quad (2.1)$$

where

$$\mathcal{L}^{\mathrm{eff}}(A,b) = -\mathrm{i}\mathrm{tr}\langle x|\ln(\mathrm{i}\hat{\partial}-\hat{A}+\hat{b}\gamma^5-m)|x\rangle \quad (2.2)$$

is the effective lagrangian, and the trace over spinor indices is denoted by tr. In the most general case (i.e. for arbitrary space-time dependent vectors $A^\mu(x)$ and $b^\mu(x)$), the effective action can be represented as a sum of two terms

$$\Gamma(A,b) = -\frac{\mathrm{i}}{2}\mathrm{Tr}\ln\mathcal{H} - \mathrm{Tr}\arctan[(\hat{\partial}+\mathrm{i}\hat{A}-\mathrm{i}\hat{b}\gamma^5)m^{-1}], \quad (2.3)$$

where

$$\mathcal{H} = -(\mathrm{i}\hat{\partial}-\hat{A}+\hat{b}\gamma^5)^2+m^2, \quad (2.4)$$

and Tr is the trace of the differential operator in functional space: $\mathrm{Tr}U = \int \mathrm{d}^4x\mathrm{tr}\langle x|U|x\rangle$. Since the trace of an odd number of γ-matrices vanishes, one gets the relation

$$\begin{aligned} &\frac{\delta}{\delta A_\mu(x)}\mathrm{Tr}\arctan[(\hat{\partial}+\mathrm{i}\hat{A}-\mathrm{i}\hat{b}\gamma^5)m^{-1}] \\ &= \frac{\delta}{\delta b_\mu(x)}\mathrm{Tr}\arctan[(\hat{\partial}+\mathrm{i}\hat{A}-\mathrm{i}\hat{b}\gamma^5)m^{-1}] = 0, \end{aligned} \quad (2.5)$$

and, therefore, the second term in the right-hand side of (2.3) can be neglected being inessential. As to the first term in the right-hand side of (2.3), it can be related to the zeta function of the operator $\mathcal{H}$ [24–26]

$$-\frac{\mathrm{i}}{2}\mathrm{Tr}\ln\mathcal{H} = \frac{\mathrm{i}}{2}\left(\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{Tr}\mathcal{H}^{-z}\right)\Big|_{z=0}. \quad (2.6)$$

Using an integral representation for the zeta function,

$$\mathrm{Tr}\mathcal{H}^{-z} = \frac{1}{\Gamma(z)}\int_0^\infty \mathrm{d}\tau\tau^{z-1}\mathrm{Tr}\mathrm{e}^{-\tau\mathcal{H}}, \quad \mathrm{Re}z>0, \quad (2.7)$$

where $\Gamma(z)$ is the Euler gamma function, one gets an integral representation for the effective action, thus:

$$\Gamma(A,b) = \frac{\mathrm{i}}{2}\int_0^\infty \frac{\mathrm{d}\tau}{\tau}\mathrm{Tr}\mathrm{e}^{-\tau\mathcal{H}}. \quad (2.8)$$

Taking functional derivatives of (2.3), let us define the vector current

$$J^\mu(x) \equiv -\frac{\delta\Gamma(A,b)}{\delta A_\mu(x)} = \mathrm{i}\mathrm{tr}\gamma^\mu G(x,x), \quad (2.9)$$

and the axial-vector current

$$J^{\mu5}(x) \equiv \frac{\delta\Gamma(A,b)}{\delta b_\mu(x)} = \mathrm{i}\mathrm{tr}\gamma^\mu\gamma^5 G(x,x), \quad (2.10)$$

where

$$G(x,y) = \langle x|(\mathrm{i}\hat{\partial}-\hat{A}+\hat{b}\gamma^5+m)\mathcal{H}^{-1}|y\rangle \quad (2.11)$$

is the Green's function. One can write an integral representation for the latter:

$$G(x,y) = \int_0^\infty \mathrm{d}\tau\langle x|(\mathrm{i}\hat{\partial}-\hat{A}+\hat{b}\gamma^5+m)\mathrm{e}^{-\tau\mathcal{H}}|y\rangle. \quad (2.12)$$

In the case when b_μ is space-time independent, the effective action can be represented in the form

$$\Gamma(A,b) = \Gamma(A,0) + \int_0^1 \mathrm{d}u\int \mathrm{d}^4x b_\mu J^{\mu5}(x;u), \quad (2.13)$$

where $J^{\mu5}(x;u)$ is the axial-vector current with ub substituted for b. It should be emphasized that most of the above relations are purely formal, since they suffer from ultraviolet divergencies. For instance, the Green's function (2.11) is well-defined at $x \neq y$, and diverges at $x \to y$. To regularize the divergence, one introduces a cut-off at the lower limit of the integral in the representation (2.12). In this way one gets a regularized definition of the currents (2.9) and (2.10), which, after appropriate integration, yield the regularized expression for the effective action.

In the present paper we restrict ourselves to the case of the space-time independent field tensor $F^{\mu\nu}$ and vector b^μ. Then the operator $\mathcal{H}$ (2.4) takes the form

$$\mathcal{H} = -\pi^\mu\pi_\mu - 2\mathrm{i}\gamma^5\sigma^{\mu\nu}b_\mu\pi_\nu + \frac{1}{2}F_{\mu\nu}\sigma^{\mu\nu} + b^\mu b_\mu + m^2, \quad (2.14)$$

where

$$\pi_\mu = \mathrm{i}\partial_\mu - A_\mu, \qquad \sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu,\gamma^\nu]_-. \quad (2.15)$$

Our aim is to find the currents (2.9) and (2.10) and then the effective lagrangian (2.2).

3 Proper-time method

Rotating the integration path in (2.12) by an angle $\pi/2$ in the anticlockwise direction (i.e. substituting τ by is), we represent the Green's function in the form

$$G(x,y) = \mathrm{i}\int_0^\infty \mathrm{d}s\langle x|(\hat{\pi}+\hat{b}\gamma^5+m)\mathrm{e}^{-\mathrm{i}s\mathcal{H}}|y\rangle, \quad (3.1)$$

where it is implied that the mass squared in $\mathcal{H}$ entails a small negative imaginary part, $m^2 \to m^2 - \mathrm{i}\epsilon$. The idea of the proper-time method of Fock [22] and Schwinger [23] is to treat the operator $\mathcal{H}$ as a Hamilton operator that governs evolution in the "time" s of a hypothetical quantum mechanical system. Then the transition amplitude (the matrix element of the evolution operator $\exp(-\mathrm{i}s\mathcal{H})$),

$$\langle x|\mathrm{e}^{-\mathrm{i}s\mathcal{H}}|y\rangle \equiv \langle x(s)|y(0)\rangle, \tag{3.2}$$

where

$$|y(0)\rangle = |y\rangle, \qquad |x(s)\rangle = \mathrm{e}^{\mathrm{i}s\mathcal{H}}|x\rangle, \tag{3.3}$$

satisfy the evolution equation

$$\mathrm{i}\partial_s\langle x(s)|y(0)\rangle = \langle x(s)|\mathcal{H}|y(0)\rangle, \tag{3.4}$$

with boundary conditions

$$\lim_{s\to 0}\langle x(s)|y(0)\rangle = \delta(x-y), \quad \lim_{s\to\infty}\langle x(s)|y(0)\rangle = 0. \tag{3.5}$$

The commutation relations

$$\begin{aligned} &[x^\mu,\pi^\nu]_- = -\mathrm{i}g^{\mu\nu}, \qquad [\pi^\mu,\pi^\nu]_- = -\mathrm{i}F^{\mu\nu},\\ &[\sigma^{\mu\nu},\sigma^{\omega\rho}]_- = 2\mathrm{i}(\sigma^{\mu\rho}g^{\nu\omega} - \sigma^{\nu\rho}g^{\mu\omega} - \sigma^{\mu\omega}g^{\nu\rho} + \sigma^{\nu\omega}g^{\mu\rho}),\\ &[x^\mu,x^\nu]_- = [x^\mu,\sigma^{\omega\rho}]_- = [\pi^\mu,\sigma^{\omega\rho}]_- = 0, \end{aligned} \tag{3.6}$$

are invariant under the unitary transformation

$$\begin{aligned} X^\mu(s) &= \mathrm{e}^{\mathrm{i}s\mathcal{H}}X^\mu(0)\mathrm{e}^{-\mathrm{i}s\mathcal{H}},\\ \Pi^\mu(s) &= \mathrm{e}^{\mathrm{i}s\mathcal{H}}\Pi^\mu(0)\mathrm{e}^{-\mathrm{i}s\mathcal{H}},\\ \Sigma^{\mu\nu}(s) &= \mathrm{e}^{\mathrm{i}s\mathcal{H}}\Sigma^{\mu\nu}(0)\mathrm{e}^{-\mathrm{i}s\mathcal{H}}, \end{aligned} \tag{3.7}$$

where

$$X^\mu(0) = x^\mu, \quad \Pi^\mu(0) = \pi^\mu, \quad \Sigma^{\mu\nu}(0) = \sigma^{\mu\nu}, \tag{3.8}$$

can be regarded as quantum mechanical observable operators in the Schrodinger representation, while $X^\mu(s)$, $\Pi^\mu(s), \Sigma^{\mu\nu}(s)$ can be considered those in the Heisenberg representation. The latter satisfy the evolution equations:

$$\begin{aligned} &\dot X^\mu(s) = \mathrm{i}[\mathcal{H}, X^\mu(s)]_-, \qquad \dot\Pi^\mu(s) = \mathrm{i}[\mathcal{H},\Pi^\mu(s)]_-,\\ &\dot\Sigma^{\mu\nu}(s) = \mathrm{i}[\mathcal{H},\Sigma^{\mu\nu}(s)]_-. \end{aligned} \tag{3.9}$$

Using the explicit form of the Hamilton operator $\mathcal{H}$ and the commutation relations (3.6), one can compute the commutators on the right hand sides of (3.9), and then solve this system of equations. Using the solution, one can compute the matrix element on the right hand side of (3.4), and then solve this equation and find the transition amplitude (3.2). The consistency relations must hold:

$$\begin{aligned} \langle x(s)|\Pi_\mu(s)|y(0)\rangle &= [\mathrm{i}\partial_\mu^{(x)} - A_\mu(x)]\langle x(s)|y(0)\rangle,\\ \langle x(s)|\Pi_\mu(0)|y(0)\rangle &= [-\mathrm{i}\partial_\mu^{(y)} - A_\mu(y)]\langle x(s)|y(0)\rangle,\\ \langle x(s)|X^\mu(s)|y(0)\rangle &= x^\mu\langle x(s)|y(0)\rangle,\\ \langle x(s)|X^\mu(0)|y(0)\rangle &= y^\mu\langle x(s)|y(0)\rangle,\\ \langle x(s)|\Sigma^{\mu\nu}(s)|y(0)\rangle &= \sigma^{\mu\nu}\langle x(s)|y(0)\rangle,\\ \langle x(s)|\Sigma^{\mu\nu}(0)|y(0)\rangle &= \langle x(s)|y(0)\rangle\sigma^{\mu\nu}. \end{aligned} \tag{3.10}$$

Let us find the transition amplitude in the case of $\mathcal{H}$ given by (2.14) with constant uniform electromagnetic field strength. The system of evolution equations (3.9) takes the form

$$\begin{aligned} \dot X^\mu(s) &= 2\Pi^\mu(s) - 2\mathrm{i}\Sigma^{\mu\nu}(s)b_\nu\gamma^5,\\ \dot\Pi_\mu(s) &= 2F_{\mu\nu}\Pi^\nu(s) - 2\mathrm{i}F_{\mu\nu}\Sigma^{\nu\omega}(s)b_\omega\gamma^5,\\ \dot\Sigma^{\mu\nu}(s) &= 2[F^{\mu\omega}g_{\omega\rho}\Sigma^{\rho\nu}(s) - \Sigma^{\mu\omega}(s)g_{\omega\rho}F^{\rho\nu}]\\ &\quad + 4\mathrm{i}\Big\{[\Pi^\mu(s)b_\omega - b^\mu\Pi_\omega(s)]\Sigma^{\omega\nu}(s)\\ &\quad -\Sigma^{\mu\omega}(s)[\Pi_\omega(s)b^\nu - b_\omega\Pi^\nu(s)]\Big\}\gamma^5. \end{aligned} \tag{3.11}$$

The system is solved in the approximation linear in b, yielding, in obvious matrix notation,

$$\begin{aligned} X(s) - X(0) &= 2\mathrm{e}^{Fs}\frac{\sinh(Fs)}{F}\Pi(0)\\ &\quad - 2\mathrm{i}\mathrm{e}^{2Fs}\Sigma(0)\frac{\sinh(Fs)}{F}\mathrm{e}^{-Fs}b\gamma^5,\\ \Pi(s) &= \mathrm{e}^{2Fs}\Pi(0)\\ &\quad - 2\mathrm{i}\mathrm{e}^{2Fs}F\Sigma(0)\frac{\sinh(Fs)}{F}\mathrm{e}^{-Fs}b\gamma^5,\\ \Sigma(s) &= \mathrm{e}^{2Fs}\Sigma(0)\mathrm{e}^{-2Fs}\\ &\quad + 4\mathrm{i}\mathrm{e}^{2Fs}\Bigg\{\Bigg[\Pi(0)b\mathrm{e}^{Fs}\frac{\sinh(Fs)}{F}\\ &\quad - \frac{\sinh(Fs)}{F}\mathrm{e}^{-Fs}b\Pi(0)\Bigg]\Sigma(0)\\ &\quad - \Sigma(0)\Bigg[\Pi(0)b\mathrm{e}^{Fs}\frac{\sinh(Fs)}{F}\\ &\quad - \frac{\sinh(Fs)}{F}\mathrm{e}^{-Fs}b\Pi(0)\Bigg]\Bigg\}\mathrm{e}^{-2Fs}\gamma^5. \end{aligned} \tag{3.12}$$

Using the two last relations in (3.12), we get

$$\begin{aligned} \mathcal{H} &= -\Pi^2(s) + 2\mathrm{i}\gamma^5\Pi(s)\Sigma(s)b - \frac{1}{2}\mathrm{Sp}F\Sigma(s) + m^2\\ &= -\Pi^2(0) + 2\mathrm{i}\gamma^5\Pi(0)\Sigma(0)b\\ &\quad - \frac{1}{2}\mathrm{Sp}F\Sigma(0) + m^2. \end{aligned} \tag{3.13}$$

The last equation states that the Hamilton operator in the Heisenberg representation coincides with the one in the Schrodinger representation being independent of s, as it should. However, the matrix element of $\mathcal{H}$ on the right hand side of (3.4) is s-dependent, and, to find this dependence, one has to express the operator (3.13) through the operators $X(s), X(0)$ and either $\Sigma(s)$ or $\Sigma(0)$. Using the first relation in (3.12), we get

$$\begin{aligned} \Pi(0) &= \frac{\mathrm{e}^{-Fs}F}{2\sinh(Fs)}[X(s) - X(0)]\\ &\quad + \mathrm{i}\mathrm{e}^{Fs}\frac{F}{\sinh(Fs)}\Sigma(0)\frac{\sinh(Fs)}{F}\mathrm{e}^{-Fs}b\gamma^5,\\ \Pi(s) &= \frac{\mathrm{e}^{Fs}F}{2\sinh(Fs)}[X(s) - X(0)]\\ &\quad + \mathrm{i}\mathrm{e}^{-Fs}\frac{F}{\sinh(Fs)}\Sigma(s)\frac{\sinh(Fs)}{F}\mathrm{e}^{Fs}b\gamma^5, \end{aligned} \tag{3.14}$$

and, consequently,

$$\Pi^2(s) = \frac{1}{4}[X(s) - X(0)]\frac{F^2}{\sinh^2(Fs)}[X(s) - X(0)]$$

$$+\,\mathrm{i}[X(s)-X(0)]\frac{\mathrm{e}^{-2Fs}F^2}{\sinh^2(Fs)}\Sigma(s)\frac{\sinh(Fs)}{F}\mathrm{e}^{Fs}b\gamma^5. \tag{3.15}$$

Using the commutational relation

$$[X_\mu(s),X_\nu(0)]_-=\mathrm{i}\left(\frac{\mathrm{e}^{Fs}\sinh(Fs)}{F}\right)_{\mu\nu}, \tag{3.16}$$

and adding relevant terms to $-\Pi^2(s)$, we get $\mathcal{H}$ in (3.13) as a proper-time-ordered function of $X(s)$ and $X(0)$:

$$\begin{aligned}\mathcal{H}=&-\frac{1}{4}X(s)\frac{F^2}{\sinh^2(Fs)}X(s)+\frac{1}{2}X(s)\frac{F^2}{\sinh^2(Fs)}X(0)\\&-\frac{1}{4}X(0)\frac{F^2}{\sinh^2(Fs)}X(0)-\frac{\mathrm{i}}{2}\mathrm{Sp}F\coth(Fs)\\&-\frac{1}{2}\mathrm{Sp}F\Sigma(s)+m^2+\mathrm{i}[X(s)-X(0)]\frac{\mathrm{e}^{-Fs}F}{\sinh(Fs)}\\&\times\left[\Sigma(s)-\frac{\mathrm{e}^{-Fs}F}{\sinh(Fs)}\Sigma(s)\frac{\sinh(Fs)}{F}\mathrm{e}^{Fs}\right]b\gamma^5,\end{aligned} \tag{3.17a}$$

or, alternatively

$$\begin{aligned}\mathcal{H}=&-\frac{1}{4}X(s)\frac{F^2}{\sinh^2(Fs)}X(s)+\frac{1}{2}X(s)\frac{F^2}{\sinh^2(Fs)}X(0)\\&-\frac{1}{4}X(0)\frac{F^2}{\sinh^2(Fs)}X(0)\\&-\frac{\mathrm{i}}{2}\mathrm{Sp}F\coth(Fs)-\frac{1}{2}\mathrm{Sp}F\Sigma(0)+m^2\\&-\mathrm{i}\gamma^5b\left[\Sigma(0)-\frac{\mathrm{e}^{Fs}F}{\sinh(Fs)}\Sigma(0)\frac{\sinh(Fs)}{F}\mathrm{e}^{-Fs}\right]\\&\times\frac{\mathrm{e}^{-Fs}F}{\sinh(Fs)}[X(s)-X(0)],\end{aligned} \tag{3.17b}$$

Using the last four relations in (3.10), we find two forms of the matrix element of $\mathcal{H}$:

$$\begin{aligned}\langle x(s)|\mathcal{H}|y(0)\rangle&=P^{(a)}(x,y;s)\langle x(s)|y(0)\rangle\\&=\langle x(s)|y(0)\rangle P^{(b)}(x,y;s),\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}P^{(a)}(x,y;s)=&-\frac{1}{4}(x-y)\frac{F^2}{\sinh^2(Fs)}(x-y)\\&-\frac{\mathrm{i}}{2}\mathrm{Sp}F\coth(Fs)-\frac{1}{2}\mathrm{Sp}F\sigma+m^2\\&+\mathrm{i}(x-y)\frac{\mathrm{e}^{-Fs}F}{\sinh(Fs)}\left[\sigma-\frac{\mathrm{e}^{-Fs}F}{\sinh(Fs)}\sigma\frac{\sinh(Fs)}{F}\mathrm{e}^{Fs}\right]b\gamma^5,\end{aligned} \tag{3.19a}$$

and

$$\begin{aligned}P^{(b)}(x,y;s)=&-\frac{1}{4}(x-y)\frac{F^2}{\sinh^2(Fs)}(x-y)\\&-\frac{\mathrm{i}}{2}\mathrm{Sp}F\coth(Fs)-\frac{1}{2}\mathrm{Sp}F\sigma+m^2\\&-\mathrm{i}\gamma^5b\left[\sigma-\mathrm{e}^{Fs}\frac{\sinh(Fs)}{F}\sigma\frac{F\mathrm{e}^{-Fs}}{\sinh(Fs)}\right]\frac{\mathrm{e}^{-Fs}F}{\sinh(Fs)}(x-y).\end{aligned} \tag{3.19b}$$

Substituting (3.19) into the right hand side of the evolution equation (3.4), and solving the latter, we get two equivalent expressions for the transition amplitude:

$$\begin{aligned}\langle x(s)|y(0)\rangle=&-\frac{\mathrm{i}}{(4\pi)^2}\\&\times\exp\left\{-\mathrm{i}\int_y^x\mathrm{d}\xi\left[A+\frac{1}{2}F(\xi-y)\right]\right\}\frac{1}{s^2}\\&\times\exp\left[-\frac{\mathrm{i}}{4}(x-y)F\coth(Fs)(x-y)\right.\\&\left.-\frac{1}{2}\mathrm{Sp}\ln\frac{\sinh(Fs)}{Fs}-\mathrm{i}sm^2\right]\\&\times\exp\left[(x-y)\mathrm{e}^{-Fs}\frac{F}{\sinh(Fs)}\sigma\frac{\sinh(Fs)}{F}\mathrm{e}^{Fs}b\gamma^5\right]\\&\times\exp\left(\frac{\mathrm{i}s}{2}\mathrm{Sp}F\sigma\right),\end{aligned} \tag{3.20a}$$

and

$$\begin{aligned}\langle x(s)|y(0)\rangle=&-\frac{\mathrm{i}}{(4\pi)^2}\\&\times\exp\left\{-\mathrm{i}\int_y^x\mathrm{d}\xi\left[A+\frac{1}{2}F(\xi-y)\right]\right\}\frac{1}{s^2}\\&\times\exp\left[-\frac{\mathrm{i}}{4}(x-y)F\coth(Fs)(x-y)\right.\\&\left.-\frac{1}{2}\mathrm{Sp}\ln\frac{\sinh(Fs)}{Fs}-\mathrm{i}sm^2\right]\\&\times\exp\left(\frac{\mathrm{i}s}{2}\mathrm{Sp}F\sigma\right)\\&\times\exp\left[-\gamma^5b\mathrm{e}^{Fs}\frac{\sinh(Fs)}{F}\sigma\frac{F}{\sinh(Fs)}\mathrm{e}^{-Fs}(x-y)\right],\end{aligned} \tag{3.20b}$$

where the b dependent exponential should be understood as expanded up to the first order in b. Also we get the relations

$$\begin{aligned}&\gamma^\mu\langle x(s)|\Pi_\mu(s)|y(0)\rangle\\&=\left[\frac{1}{2}\gamma\mathrm{e}^{Fs}\frac{F}{\sinh(Fs)}(x-y)\right.\\&\left.+\,\mathrm{i}\gamma\mathrm{e}^{-Fs}\frac{F}{\sinh(Fs)}\sigma\frac{\sinh(Fs)}{F}\mathrm{e}^{Fs}b\gamma^5\right]\\&\times\langle x(s)|y(0)\rangle,\end{aligned} \tag{3.21a}$$

and

$$\begin{aligned}&\langle x(s)|\Pi_\mu(0)|y(0)\rangle\gamma^\mu\\&=\langle x(s)|y(0)\rangle\left[\frac{1}{2}(x-y)\mathrm{e}^{Fs}\frac{F}{\sinh(Fs)}\gamma\right.\\&\left.-\mathrm{i}\gamma^5b\mathrm{e}^{Fs}\frac{\sinh(Fs)}{F}\sigma\frac{F}{\sinh(Fs)}\mathrm{e}^{-Fs}\gamma\right],\end{aligned} \tag{3.21b}$$

and one can verify that the first two relations in (3.10) are valid. Consequently, we get two equivalent representations for Green's function $G(x,y)$ which, after rotating the integration path in (3.1) back ($s = -\mathrm{i}\tau$), take the form

$$G(x,y) = \int_0^\infty \mathrm{d}\tau \left[\frac{\mathrm{i}}{2}\gamma \mathrm{e}^{-\mathrm{i}F\tau}\frac{F}{\sin(F\tau)}(x-y) + \mathrm{i}\gamma \mathrm{e}^{\mathrm{i}F\tau}\frac{F}{\sin(F\tau)}\sigma\frac{\sin(F\tau)}{F}\mathrm{e}^{-\mathrm{i}F\tau}b\gamma^5 + \gamma b\gamma^5 + m\right] \times \langle x(-\mathrm{i}\tau)|y(0)\rangle, \tag{3.22a}$$

and

$$G(x,y) = \int_0^\infty \mathrm{d}\tau \langle x(-\mathrm{i}\tau)|y(0)\rangle \left[\frac{\mathrm{i}}{2}(x-y)\frac{F}{\sin(F\tau)}\mathrm{e}^{-\mathrm{i}F\tau}\gamma - \mathrm{i}\gamma^5 b \mathrm{e}^{-\mathrm{i}F\tau}\frac{\sin(F\tau)}{F}\sigma\frac{F}{\sin(F\tau)}\mathrm{e}^{\mathrm{i}F\tau}\gamma - \gamma^5 b\gamma + m\right]. \tag{3.22b}$$

4 Currents

Inserting either (3.20a) or (3.20b) into (3.22a) and (3.22b), taking the limit $y \to x$ and retaining terms which are not higher than first order in b^μ, we get two equivalent expressions for the vector current (2.9):

$$J^\mu = -\frac{1}{(4\pi)^2}\int_0^\infty \frac{\mathrm{d}\tau}{\tau^2}\exp\left[-\tau m^2 - \frac{1}{2}\mathrm{Sp}\ln\frac{\sin(F\tau)}{F\tau}\right] \times \mathrm{tr}\gamma^5\gamma^\mu\gamma\left[\mathrm{i}\mathrm{e}^{\mathrm{i}F\tau}\frac{F}{\sin(F\tau)}\sigma\frac{\sin(F\tau)}{F}\mathrm{e}^{-\mathrm{i}F\tau} + 1\right] b \times \exp\left(\frac{\tau}{2}\mathrm{Sp}\sigma F\right), \tag{4.1a}$$

and

$$J'^\mu = \frac{1}{(4\pi)^2}\int_0^\infty \frac{\mathrm{d}\tau}{\tau^2}\exp\left[-\tau m^2 - \frac{1}{2}\mathrm{Sp}\ln\frac{\sin(F\tau)}{F\tau}\right] \times \mathrm{tr}\exp\left(\frac{\tau}{2}\mathrm{Sp}\sigma F\right) \times b\left[\mathrm{i}\mathrm{e}^{-\mathrm{i}F\tau}\frac{\sin(F\tau)}{F}\sigma\frac{F}{\sin(F\tau)}\mathrm{e}^{\mathrm{i}F\tau} + 1\right]\gamma\gamma^\mu\gamma^5, \tag{4.1b}$$

and for the axial-vector current (2.10):

$$J^{\mu 5} = \frac{1}{(4\pi)^2}\int_0^\infty \frac{\mathrm{d}\tau}{\tau^2}\exp\left[-\tau m^2 - \frac{1}{2}\mathrm{Sp}\ln\frac{\sin(F\tau)}{F\tau}\right] \times \mathrm{tr}\gamma^\mu\gamma\left[\mathrm{i}\mathrm{e}^{\mathrm{i}F\tau}\frac{F}{\sin(F\tau)}\sigma\frac{\sin(F\tau)}{F}\mathrm{e}^{-\mathrm{i}F\tau} + 1\right] b \times \exp\left(\frac{\tau}{2}\mathrm{Sp}\sigma F\right), \tag{4.2a}$$

and

$$J'^{\mu 5} = \frac{1}{(4\pi)^2}\int_0^\infty \frac{\mathrm{d}\tau}{\tau^2}\exp\left[-\tau m^2 - \frac{1}{2}\mathrm{Sp}\ln\frac{\sin(F\tau)}{F\tau}\right] \times \mathrm{tr}\exp\left(\frac{\tau}{2}\mathrm{Sp}\sigma F\right) \times b\left[\mathrm{i}\mathrm{e}^{-\mathrm{i}F\tau}\frac{\sin(F\tau)}{F}\sigma\frac{F}{\sin(F\tau)}\mathrm{e}^{\mathrm{i}F\tau} + 1\right]\gamma\gamma^\mu. \tag{4.2b}$$

One can notice that

$$(J'^\mu)^* = J^\mu, \qquad (J'^{\mu 5})^* = J^{\mu 5}, \tag{4.3}$$

and, therefore, both vector and axial-vector currents have to be real, if the representations (3.22a) and (3.22b) are indeed equivalent. In order to take traces over the γ-matrices in (4.1) and (4.2), one uses an expansion of the exponential of the σ-matrix,

$$\exp\left(\frac{\tau}{2}\mathrm{Sp}\sigma F\right) = C_1(\tau)I + C_2(\tau)\mathrm{Sp}\sigma F + C_3(\tau)\mathrm{i}\gamma^5 + C_4(\tau)\mathrm{i}\gamma^5\mathrm{Sp}\sigma F, \tag{4.4}$$

and the relations

$$\begin{aligned}(\tilde{F}^2)_{\mu\nu} &= (F^2)_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathrm{Sp}F^2,\\ (F\tilde{F})_{\mu\nu} &= (\tilde{F}F)_{\mu\nu} = \frac{1}{4}g_{\mu\nu}\mathrm{Sp}(F\tilde{F}),\\ \varepsilon^{\mu\nu\alpha\beta}(FK)_{\alpha\beta} &= -(\tilde{F}K)^{\mu\nu} - (K\tilde{F})^{\mu\nu} + \tilde{F}^{\mu\nu}\mathrm{Sp}K,\end{aligned} \tag{4.5}$$

where $K_{\mu\nu}$ is an arbitrary symmetric second-rank Lorentz tensor, and we have the evident identities $\mathrm{tr}\gamma^5\gamma^\mu\gamma^\nu\sigma^{\alpha\beta} = -4\varepsilon^{\mu\nu\alpha\beta}$, $\mathrm{tr}\gamma^\mu\gamma^\nu\sigma^{\alpha\beta} = -4\mathrm{i}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})$. Thus, we get

$$\begin{aligned}\mathrm{Re}J^\mu &= \frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}\tau \mathrm{e}^{-\tau m^2}\omega^{\mu\nu}(\tau)b_\nu,\\ \mathrm{Im}J^\mu &= \frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}\tau \mathrm{e}^{-\tau m^2}\rho^{\mu\nu}(\tau)b_\nu,\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\mathrm{Re}J^{\mu 5} &= \frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}\tau \mathrm{e}^{-\tau m^2}\omega_5^{\mu\nu}(\tau)b_\nu,\\ \mathrm{Im}J^{\mu 5} &= \frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}\tau \mathrm{e}^{-\tau m^2}\rho_5^{\mu\nu}(\tau)b_\nu,\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}\omega^{\mu\nu}(\tau) = -C_0(\tau)\Bigg\{ & C_1(\tau)\left(\tilde{F}\frac{\sin 2F\tau}{F}\right)^{\mu\nu}\\ & + C_2(\tau)\left[4\tilde{F}^{\mu\nu} - \left(\tilde{F}\frac{\sin 2F\tau}{F}\right)^{\mu\nu}\mathrm{Sp}F\cot F\tau\right]\\ & + C_3(\tau)\left[(\sin 2F\tau)^{\mu\nu} - \left(\frac{\sin^2 F\tau}{F}\right)^{\mu\nu}\mathrm{Sp}F\cot F\tau\right]\\ & - C_4(\tau)\Bigg[4(F\cos 2F\tau)^{\mu\nu} - (\sin 2F\tau)^{\mu\nu}\mathrm{Sp}F\cot F\tau\\ & + 2\left(\frac{\sin^2 F\tau}{F}\right)^{\mu\nu}\mathrm{Sp}F^2\Bigg]\Bigg\},\end{aligned} \tag{4.8}$$

$$\rho^{\mu\nu}(\tau) = C_0(\tau)\left\{2C_1(\tau)\left(\tilde{F}\frac{\sin^2 F\tau}{F}\right)^{\mu\nu} - 2C_2(\tau)\left(\tilde{F}\frac{\sin^2 F\tau}{F}\right)^{\mu\nu}\mathrm{Sp}F\cot F\tau - C_3(\tau)\left[2(\cos^2 F\tau)^{\mu\nu} - \frac{1}{2}\left(\frac{\sin 2F\tau}{F}\right)^{\mu\nu}\mathrm{Sp}F\cot F\tau\right] - C_4(\tau)\left[4(F\sin 2F\tau)^{\mu\nu} - 2(\sin^2 F\tau)^{\mu\nu}\mathrm{Sp}F\cot F\tau - \left(\frac{\sin 2F\tau}{F}\right)^{\mu\nu}\mathrm{Sp}F^2\right]\right\}, \quad (4.9)$$

$$\omega_5^{\mu\nu}(\tau) = C_0(\tau) \times \left\{C_1(\tau)\left[2(\cos^2 F\tau)^{\mu\nu} - \frac{1}{2}\left(\frac{\sin 2F\tau}{F}\right)^{\mu\nu}\mathrm{Sp}F\cot F\tau\right] + C_2(\tau)\left[4(F\sin 2F\tau)^{\mu\nu} - 2(\sin^2 F\tau)^{\mu\nu}\mathrm{Sp}F\cot F\tau - \left(\frac{\sin 2F\tau}{F}\right)^{\mu\nu}\mathrm{Sp}F^2\right] + 2C_3(\tau)\left(\tilde{F}\frac{\sin^2 F\tau}{F}\right)^{\mu\nu} - 2C_4(\tau)\left(\tilde{F}\frac{\sin^2 F\tau}{F}\right)^{\mu\nu}\mathrm{Sp}F\cot(F\tau)\right\}, \quad (4.10)$$

$$\rho_5^{\mu\nu}(\tau) = -C_0(\tau) \times \left\{C_1(\tau)\left[(\sin 2F\tau)^{\mu\nu} - \left(\frac{\sin^2 F\tau}{F}\right)^{\mu\nu}\mathrm{Sp}F\cot F\tau\right] - C_2(\tau)\left[4(F\cos 2F\tau)^{\mu\nu} - (\sin 2F\tau)^{\mu\nu}\mathrm{Sp}F\cot F\tau + 2\left(\frac{\sin^2 F\tau}{F}\right)^{\mu\nu}\mathrm{Sp}F^2\right] - C_3(\tau)\left(\tilde{F}\frac{\sin 2F\tau}{F}\right)^{\mu\nu} - C_4(\tau)\left[4\tilde{F}^{\mu\nu} - \left(\tilde{F}\frac{\sin 2F\tau}{F}\right)^{\mu\nu}\mathrm{Sp}F\cot F\tau\right]\right\}, \quad (4.11)$$

and the notation

$$C_0(\tau) = \tau^{-2}\exp\left(-\frac{1}{2}\mathrm{Sp}\ln\frac{\sin F\tau}{F\tau}\right) \quad (4.12)$$

is introduced for brevity. The coefficient functions $C_j(\tau)$ ($j = \overline{1,4}$) are given explicitly by the expressions

$$C_1(\tau) = \mathrm{Re}\cosh\left[\tau\sqrt{\frac{1}{2}(-\mathrm{Sp}F^2 + \mathrm{i}\mathrm{Sp}F\tilde{F})}\right],$$

$$C_2(\tau) = \mathrm{Re}\frac{\sinh\left[\tau\sqrt{\frac{1}{2}(-\mathrm{Sp}F^2 + \mathrm{i}\mathrm{Sp}F\tilde{F})}\right]}{\sqrt{2(-\mathrm{Sp}F^2 + \mathrm{i}\mathrm{Sp}F\tilde{F})}},$$

$$C_3(\tau) = \mathrm{Im}\cosh\left[\tau\sqrt{\frac{1}{2}(-\mathrm{Sp}F^2 + \mathrm{i}\mathrm{Sp}F\tilde{F})}\right],$$

$$C_4(\tau) = \mathrm{Im}\frac{\sinh\left[\tau\sqrt{\frac{1}{2}(-\mathrm{Sp}F^2 + \mathrm{i}\mathrm{Sp}F\tilde{F})}\right]}{\sqrt{2(-\mathrm{Sp}F^2 + \mathrm{i}\mathrm{Sp}F\tilde{F})}}, \quad (4.13)$$

and $C_0(\tau)$ (4.12) is expressed as [23]

$$C_0(\tau) = \mathrm{Sp}F\tilde{F}[4C_3(\tau)]^{-1}. \quad (4.14)$$

There are remarkable relations among the $C_j(\tau)$:

$$C_1(\tau)(\tan F\tau)^{\mu\nu} = 2[C_2(\tau)F^{\mu\nu} + C_4(\tau)\tilde{F}^{\mu\nu}] \quad (4.15)$$

and

$$C_3(\tau)(\cot F\tau)^{\mu\nu} = 2[C_2(\tau)\tilde{F}^{\mu\nu} - C_4(\tau)F^{\mu\nu}]. \quad (4.16)$$

In the appendix we prove these relations and find that

$$\omega^{\mu\nu}(\tau) = \rho^{\mu\nu}(\tau) = \rho_5^{\mu\nu}(\tau) = 0, \quad (4.17)$$

and (see (4.18) on top of the next page). Thus, the vector current J^μ (2.9) is vanishing, whereas the axial-vector current $J^{\mu 5}$ (2.10) is real (as it should) and divergent. Regularizing the divergence by introducing a small positive τ_0 as the lower limit of the τ-integral in (4.7), and separating the term divergent at $\tau_0 \to 0$ from the convergent one, we get

$$J^{\mu 5} = -\frac{b^\mu}{2\pi^2}\left[\frac{1}{\tau_0} + m^2\ln(m^2\tau_0\mathrm{e}^{\gamma-1})\right] + \frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}\tau\mathrm{e}^{-\tau m^2}\left[\omega_5^{\mu\nu}(\tau) + \frac{2}{\tau^2}g^{\mu\nu}\right]b_\nu, \quad (4.19)$$

where γ is the Euler constant.

5 Effective lagrangian

Using either (3.20a) or (3.20b), we get

$$\langle x(-\mathrm{i}\tau)|x(0)\rangle = \frac{\mathrm{i}}{(4\pi)^2}\frac{1}{\tau^2}\exp\left[-\tau m^2 - \frac{1}{2}\mathrm{Sp}\ln\frac{\sin(F\tau)}{F\tau}\right] \times \exp\left(\frac{\tau}{2}\mathrm{Sp}\sigma F\right), \quad (5.1)$$

and, consequently,

$$\frac{\mathrm{i}}{2}\int_0^\infty \frac{\mathrm{d}\tau}{\tau}\mathrm{tr}\langle x|\mathrm{e}^{-\tau\mathcal{H}}|x\rangle = -\frac{1}{8\pi^2}\int_0^\infty \frac{\mathrm{d}\tau}{\tau}\mathrm{e}^{-\tau m^2}C_0(\tau)C_1(\tau), \quad (5.2)$$

where $C_0(\tau)$ and $C_1(\tau)$ are given by (4.13) and (4.14). This coincides with the Schwinger result [23], proving that the corrections to the effective action linear in b are absent, which is consistent with the vanishing of the vector current J^μ. The non-vanishing of the axial-vector current $J^{\mu 5}$ results in the appearance of the corrections to the effective action quadratic in b. Actually, (2.13), rewritten in terms

$$\omega_5^{\mu\nu}(\tau) = -g^{\mu\nu}\left\{\mathrm{Sp}F^2 + \frac{\left[\mathrm{Sp}F^4 - \frac{1}{2}(\mathrm{Sp}F^2)^2\right](\mathrm{Sp}F\cot F\tau)^2 + 4(\mathrm{Sp}F^3\cot F\tau)^2}{2[4\mathrm{Sp}F^4 - (\mathrm{Sp}F^2)^2]}\right\}$$
$$+ (F^2)^{\mu\nu}\left[2 - \frac{\mathrm{Sp}F\cot F\tau(\mathrm{Sp}F^2\mathrm{Sp}F\cot F\tau - 4\mathrm{Sp}F^3\cot F\tau)}{4\mathrm{Sp}F^4 - (\mathrm{Sp}F^2)^2}\right]. \quad (4.18)$$

of the effective lagrangian (density of the effective action), takes the form

$$\mathcal{L}^{\text{eff}}(A,b) - \mathcal{L}^{\text{eff}}(A,0) = \frac{1}{2}b_\mu J^{\mu 5}, \quad (5.3)$$

where we have used the linearity of $J^{\mu 5}$ (4.19) in b and integrated over the parameter u. Thus, identifying $\mathcal{L}^{\text{eff}}(A,0)$ with (5.2) we get

$$\mathcal{L}^{\text{eff}}(A,b) = -\frac{1}{8\pi^2}\int_0^\infty \mathrm{d}\tau \mathrm{e}^{-\tau m^2} \times\left[\frac{1}{\tau}C_0(\tau)C_1(\tau) - b\omega_5(\tau)b\right], \quad (5.4)$$

where $\omega_5^{\mu\nu}(\tau)$ is given by (4.18) and the first term in square brackets can also be represented in a manner similar to (4.18):

$$C_0(\tau)C_1(\tau) = \frac{1}{16}\left[(\mathrm{Sp}F\cot F\tau)^2 - \frac{(\mathrm{Sp}F^2\mathrm{Sp}F\cot F\tau - 4\mathrm{Sp}F^3\cot F\tau)^2}{4\mathrm{Sp}F^4 - (\mathrm{Sp}F^2)^2}\right]. \quad (5.5)$$

Regularizing the divergence of the integral in (5.4) by introducing a cut-off τ_0 and separating the terms divergent at $\tau_0 \to 0$ from the convergent ones, we get

$$\mathcal{L}^{\text{eff}}(A,b) = \frac{1}{(4\pi)^2}\left[\frac{1}{\tau_0^2} - 2\frac{m^2}{\tau_0} - m^4\ln(m^2\tau_0\mathrm{e}^{\gamma-3/2})\right] - \frac{\mathrm{Sp}F^2}{3(4\pi)^2}\ln(m^2\tau_0\mathrm{e}^\gamma) - \frac{b^2}{(2\pi)^2}\left[\frac{1}{\tau_0} + m^2\ln(m^2\tau_0\mathrm{e}^{\gamma-1})\right] - \frac{1}{8\pi^2}\int_0^\infty \mathrm{d}\tau\mathrm{e}^{-\tau m^2}\left[\frac{1}{\tau}C_0(\tau)C_1(\tau) - \frac{1}{\tau^3} + \frac{\mathrm{Sp}F^2}{6\tau} - b\omega_5(\tau)b - 2\frac{b^2}{\tau^2}\right]. \quad (5.6)$$

Subtraction of terms in the first square brackets on the right hand side of (5.6) corresponds to the requirement that the effective lagrangian should vanish at vanishing F and b. Subtraction of other terms which are not included into the convergent integral corresponds to the redefinition (renormalization) of bare parameters of the lagrangian of the bosonic sector. Namely, the logarithmically divergent term which is proportional to $\mathrm{Sp}F^2$ is combined with the Maxwell term to produce the charge renormalization [23]:

$$\frac{\mathrm{Sp}F^2}{4e^2} \to \frac{\mathrm{Sp}F^2}{4e_{\text{ren}}^2}.$$

In a quite similar way, the terms in the second square brackets on the right hand side of (5.6) are absorbed into the renormalization of the coefficient before b^2: $-\kappa^2 b^2 \to -\kappa_{\text{ren}}^2 b^2$. Thus, we are left with the finite renormalized effective lagrangian,

$$\mathcal{L}_{\text{ren}}^{\text{eff}}(A,b) = -\frac{1}{8\pi^2}\int_0^\infty \mathrm{d}\tau\mathrm{e}^{-\tau m^2}\left[\frac{1}{\tau}C_0(\tau)C_1(\tau) - \frac{1}{\tau^3} + \frac{\mathrm{Sp}F^2}{6\tau} - b\omega_5(\tau)b - 2\frac{b^2}{\tau^2}\right], \quad (5.7)$$

which with the use of (4.18) and (5.5) is rewritten in explicit form (see (5.8) on top of the next page). In the case of a weak field strength, $F^{\mu\nu} \ll m^2$, the last expression takes the form

$$\mathcal{L}_{\text{ren}}^{\text{eff}}(A,b) = \frac{1}{12\pi^2}\left\{\frac{1}{120m^4}\left[7\mathrm{Sp}F^4 - \frac{5}{2}(\mathrm{Sp}F^2)^2\right] + \frac{1}{m^2}\left(bF^2b - \frac{1}{2}b^2\mathrm{Sp}F^2\right)\right\}; \quad (5.9)$$

note that terms of the zeroth order in b were first obtained more than 65 years ago by Heisenberg and Euler [27] and Weisskopf [28]. In the case of a purely electric or magnetic field one has $\mathrm{Sp}F^4 = (1/2)(\mathrm{Sp}F^2)^2$ and $\mathrm{Sp}F^2 = 2E^2$ or $\mathrm{Sp}F^2 = -2H^2$, where E and H are the absolute values of the electric and magnetic field strengths, correspondingly. Expression (5.8) takes the form

$$\mathcal{L}_{\text{ren}}^{\text{eff}}(A,b) = -\frac{1}{8\pi^2}\int_0^\infty \mathrm{d}\tau\mathrm{e}^{-\tau m^2} \times\left\{\frac{1}{\tau}\left(\frac{E}{\tau}\cot E\tau - \frac{1}{\tau^2} + \frac{1}{3}E^2\right) - 2[\mathbf{b}^2 - E^{-2}(\mathbf{bE})^2]\left(\frac{E^2}{\sin^2 E\tau} - \frac{1}{\tau^2}\right)\right\} \quad (5.10)$$

in the case of a purely electric field with strength $\mathbf{E}$ ($|\mathbf{E}| = E$), and

$$\mathcal{L}_{\text{ren}}^{\text{eff}}(A,b) = -\frac{1}{8\pi^2}\int_0^\infty \mathrm{d}\tau\mathrm{e}^{-\tau m^2}\left\{\frac{1}{\tau}\left(\frac{H}{\tau}\coth H\tau - \frac{1}{\tau^2} - \frac{1}{3}H^2\right) + 2[(b^0)^2 - H^{-2}(\mathbf{bH})^2]\left(\frac{H^2}{\sinh^2 H\tau} - \frac{1}{\tau^2}\right)\right\} \quad (5.11)$$

in the case of a purely magnetic field with strength $\mathbf{H}$ ($|\mathbf{H}| = H$). Note that the effective lagrangian does not depend

$$\mathcal{L}^{\rm eff}_{\rm ren}(A,b) = -\frac{1}{8\pi^2}\int_0^\infty {\rm d}\tau {\rm e}^{-\tau m^2}\left(\frac{1}{16\tau}\left[({\rm Sp}F\cot F\tau)^2 - \frac{({\rm Sp}F^2{\rm Sp}F\cot F\tau - 4{\rm Sp}F^3\cot F\tau)^2}{4{\rm Sp}F^4 - ({\rm Sp}F^2)^2}\right]\right.$$
$$-\frac{1}{\tau^3} + \frac{{\rm Sp}F^2}{6\tau} + b^2\left\{{\rm Sp}F^2 + \frac{\left[{\rm Sp}F^4 - \frac{1}{2}({\rm Sp}F^2)^2\right]({\rm Sp}F\cot F\tau)^2 + 4({\rm Sp}F^3\cot F\tau)^2}{2[4{\rm Sp}F^4 - ({\rm Sp}F^2)^2]} - \frac{2}{\tau^2}\right\}$$
$$\left. - bF^2b\left[2 - \frac{{\rm Sp}F\cot F\tau({\rm Sp}F^2{\rm Sp}F\cot F\tau - 4{\rm Sp}F^3\cot F\tau)}{4{\rm Sp}F^4 - ({\rm Sp}F^2)^2}\right]\right). \tag{5.8}$$

on the time component of b^μ in the case of a purely electric field. In the case of the vector $\mathbf{E}$ directed along the vector $\mathbf{H}$ one has ${\rm Sp}F^{2n} = 2[E^{2n} + (-1)^n H^{2n}]$, and the expression (5.8) takes the form

$$\mathcal{L}^{\rm eff}_{\rm ren}(A,b) = -\frac{1}{8\pi^2}\int_0^\infty {\rm d}\tau {\rm e}^{-\tau m^2} \times\left\{\frac{1}{\tau}\left[EH\cot E\tau\coth H\tau - \frac{1}{\tau^2} + \frac{1}{3}(E^2 - H^2)\right]\right.$$
$$+2b^2\left(\frac{H^2}{\sinh^2 H\tau} - \frac{1}{\tau^2}\right) - 2\left[\mathbf{b}^2 - \frac{(\mathbf{bE})^2 + (\mathbf{bH})^2}{E^2 + H^2}\right] \times\left.\left(\frac{E^2}{\sin^2 E\tau} - \frac{H^2}{\sinh^2 H\tau}\right)\right\}. \tag{5.12}$$

Finally, in the case of $E = H$, when ${\rm Sp}F^2 = 0$ and ${\rm Sp}F^4 = \frac{1}{4}({\rm Sp}F\tilde{F})^2 = 4H^4\cos^2\theta$ (θ is the angle between the vectors $\mathbf{E}$ and $\mathbf{H}$), expression (5.8) takes the form

$$\mathcal{L}^{\rm eff}_{\rm ren}(A,b) = -\frac{1}{8\pi^2}\int_0^\infty {\rm d}\tau {\rm e}^{-\tau m^2} \times\left\{\frac{1}{\tau}\left[H^2\cos\theta\cot(\tau H\sqrt{\cos\theta})\coth(\tau H\sqrt{\cos\theta}) - \frac{1}{\tau^2}\right]\right.$$
$$- 2b^2H^2 \times\left[\frac{\sin^2\frac{\theta}{2}}{\sin^2(\tau H\sqrt{\cos\theta})} - \frac{\cos^2\frac{\theta}{2}}{\sinh^2(\tau H\sqrt{\cos\theta})} + \frac{1}{\tau^2 H^2}\right]$$
$$- [2\mathbf{b}^2H^2 - (\mathbf{bE})^2 - (\mathbf{bH})^2] \times\left.\left[\frac{1}{\sin^2(\tau H\sqrt{\cos\theta})} - \frac{1}{\sinh^2(\tau H\sqrt{\cos\theta})}\right]\right\}. \tag{5.13}$$

6 Conclusion

In the present paper we have used the proper-time method [22,23] to calculate the effective action of the theory with the lagrangian (1.2) in the case when the electromagnetic field strength $F^{\mu\nu}$ and the vector b^μ are space-time independent. Previous attempts to solve this task [17–19] were unconvincing, because the dependence of the γ-matrices on the proper time had not been adequately taken into account. Actually, since the commutator of the Hamilton operator $\mathcal{H}$ (2.14) with σ is non-zero, the latter has to evolve in proper time as well as canonical variables do, and the correct system of the evolution equations is given by (3.11)[1]. We solve this system in the approximation linear in b and get the transition amplitude (3.20) and the Green's function (3.22). Further, we find that the vector current J^μ (2.9) is vanishing, which ensures that the Chern–Simons term is not induced, because, otherwise, the current would be non-vanishing, $J^\mu = (1/2)\tilde{F}^{\mu\nu}k_\nu$, see (1.1). Moreover, the vanishing of J^μ means that corrections to the effective action of the first order in b are absent, and parity is not violated in this approximation, although it is explicitly violated in the initial lagrangian (1.2). Also, we find that the axial-vector current $J^{\mu 5}$ (2.10) is non-vanishing and is given by the gauge invariant expressions (4.18) and (4.19). This allows us to get corrections to the effective action of the second order in b, and we find that the renormalized (finite) effective lagrangian is given by (5.8). It should be noted that the τ-integral for the terms quadratic in b in (5.8) is indeed convergent in the case of a purely magnetic field only; see (5.11). In the case of a non-vanishing electric field, when terms of the zeroth order in b develop a non-vanishing imaginary part due to simple poles of the cotangent function, the τ-integral for the terms quadratic in b is divergent due to double poles of the inverse squared sine function; see (5.10), (5.12) and (5.13). Thus, the latter τ-integral is not to be understood literally but, instead, should be regarded just as a mere algorithm to get terms up to any finite order in powers of F^2/m^4; in particular, for the lowest non-vanishing order; see (5.9). It should be emphasized that the vanishing of

[1] In the case of b^μ being equal to zero, the evolution equation for $\Sigma(s)$ decouples, and the Hamilton operator (3.13) loses the dependence on the evolution of $\Sigma(s)$, owing to the relation ${\rm Sp}F\Sigma(s) = {\rm Sp}F\Sigma(0)$ in this case

the vector current J^μ is related to the use of the approximation of the space-time independent field strength. If the field strength is inhomogeneous, then the current is non-vanishing even in the zeroth order in b. Whether the inhomogeneity of the field strength results in corrections to J^μ linear in b and, consequently, in parity violating terms in the effective action, remains an open question which needs further investigation.

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Appendix

Let us consider the quantity

$$\lambda^{\mu\nu}(\tau) = \omega^{\mu\nu} + (\rho \cot F\tau)^{\mu\nu} = -C_0\Big\{4C_2\tilde{F}^{\mu\nu} + C_3[2(\cot F\tau)^{\mu\nu} - (F^{-1})^{\mu\nu}\mathrm{Sp}F\cot F\tau] + 2C_4[2F^{\mu\nu} - (F^{-1})^{\mu\nu}\mathrm{Sp}F^2]\Big\}. \tag{A.1}$$

Since $\lambda^{\mu\nu}$ contains only odd powers of the field strength, see (4.8) and (4.9), its most general form is

$$\lambda^{\mu\nu}(\tau) = \Lambda_1(\tau)F^{\mu\nu} + \Lambda_2(\tau)\tilde{F}^{\mu\nu}, \tag{A.2}$$

where

$$\Lambda_1(\tau) = \frac{4\mathrm{Sp}F^3\lambda(\tau) - \mathrm{Sp}F^2\mathrm{Sp}F\lambda(\tau)}{(\mathrm{Sp}F^2)^2 + (\mathrm{Sp}F\tilde{F})^2},$$

$$\Lambda_2(\tau) = (\mathrm{Sp}F\tilde{F})^{-1} \times \left[\mathrm{Sp}F\lambda(\tau) - \mathrm{Sp}F^2\frac{4\mathrm{Sp}F^3\lambda(\tau) - \mathrm{Sp}F^2\mathrm{Sp}F\lambda(\tau)}{(\mathrm{Sp}F^2)^2 + (\mathrm{Sp}F\tilde{F})^2}\right]. \tag{A.3}$$

Multiplying (A.1) by appropriate powers of the field strength and taking traces, we find

$$\mathrm{Sp}F\lambda(\tau) = 2C_0\Big(C_3\mathrm{Sp}F\cot F\tau - 2C_2\mathrm{Sp}F\tilde{F} + 2C_4\mathrm{Sp}F^2\Big),$$
$$\mathrm{Sp}F^3\lambda(\tau) = -\frac{1}{2}C_0\mathrm{Sp}F\tilde{F}\Big(C_3\mathrm{Sp}\tilde{F}\cot F\tau + 2C_2\mathrm{Sp}F^2 + 2C_4\mathrm{Sp}F\tilde{F}\Big). \tag{A.4}$$

By using the eigenvalue method of Schwinger [23], one can express $\mathrm{Sp}F\cot F\tau$ and $\mathrm{Sp}\tilde{F}\cot F\tau$ via $\mathrm{Sp}F^2$ and $\mathrm{Sp}F\tilde{F}$. The eigenvalue equation for F has four solutions with eigenvalues $\pm f^{(1)}$ and $\pm f^{(2)}$, where

$$f^{(1)} = \frac{\mathrm{i}}{2\sqrt{2}} \times [(-\mathrm{Sp}F^2 + \mathrm{i}\mathrm{Sp}F\tilde{F})^{1/2} + (-\mathrm{Sp}F^2 - \mathrm{i}\mathrm{Sp}F\tilde{F})^{1/2}],$$

$$f^{(2)} = \frac{\mathrm{i}}{2\sqrt{2}} \times [(-\mathrm{Sp}F^2 + \mathrm{i}\mathrm{Sp}F\tilde{F})^{1/2} - (-\mathrm{Sp}F^2 - \mathrm{i}\mathrm{Sp}F\tilde{F})^{1/2}], \tag{A.5}$$

and the eigenvalues of $\tilde{F}$ are related to those of F:

$$\tilde{f}^{(l)} = -\frac{\mathrm{Sp}F\tilde{F}}{4f^{(l)}}, \qquad l = 1, 2. \tag{A.6}$$

Thus we get

$$\mathrm{Sp}F\cot F\tau = 2[f^{(1)}\cot(f^{(1)}\tau) + f^{(2)}\cot(f^{(2)}\tau)], \tag{A.7}$$

and

$$\mathrm{Sp}\tilde{F}\cot F\tau = -\frac{\mathrm{Sp}F\tilde{F}}{2f^{(1)}f^{(2)}} \times [f^{(2)}\cot(f^{(1)}\tau) + f^{(1)}\cot(f^{(2)}\tau)], \tag{A.8}$$

which, upon substitution of (A.5), yield

$$\mathrm{Sp}F\cot F\tau = \frac{2}{C_3}(C_2\mathrm{Sp}F\tilde{F} - C_4\mathrm{Sp}F^2) \tag{A.9}$$

and

$$\mathrm{Sp}\tilde{F}\cot F\tau = -\frac{2}{C_3}(C_2\mathrm{Sp}F^2 + C_4\mathrm{Sp}F\tilde{F}). \tag{A.10}$$

The last relations ensure that the traces in (A.4) are equal to zero, and consequently,

$$\lambda^{\mu\nu}(\tau) = 0. \tag{A.11}$$

Now, using (A.9), we can get rid of terms $(F^{-1})^{\mu\nu}$ in (A.1), and get the relation (4.16) in Sect. 4. Let us consider the quantity

$$\begin{aligned}\lambda_5^{\mu\nu}(\tau) &= \omega_5^{\mu\nu} + (\rho_5\cot F\tau)^{\mu\nu} \\ &= 2C_0\Big\{C_2[2(F\cot F\tau)^{\mu\nu} - g^{\mu\nu}\mathrm{Sp}F\cot F\tau] + C_3(\tilde{F}F^{-1})^{\mu\nu} + C_4[2(\tilde{F}\cot F\tau)^{\mu\nu} - (\tilde{F}F^{-1})^{\mu\nu}\mathrm{Sp}F\cot F\tau]\Big\} \\ &= \frac{1}{2}\left\{\frac{C_2}{C_3}\mathrm{Sp}F\tilde{F}[2(F\cot F\tau)^{\mu\nu} - g^{\mu\nu}\mathrm{Sp}F\cot F\tau] + 4(\tilde{F}^2)^{\mu\nu} + \frac{C_4}{C_3}\Big[2(F\cot F\tau)^{\mu\nu}\mathrm{Sp}F\tilde{F} - 4(\tilde{F}^2)^{\mu\nu}\mathrm{Sp}F\cot F\tau\Big]\right\},\end{aligned} \tag{A.12}$$

where in the last line (4.14) is used. Since $\lambda_5^{\mu\nu}$ contains only even powers of the field strength, see (4.10) and (4.11), its most general form is

$$\lambda_5^{\mu\nu}(\tau) = \Omega_1(\tau)g^{\mu\nu} + \Omega_2(\tau)(\tilde{F}^2)^{\mu\nu} = \left[\Omega_1(\tau) - \frac{1}{2}\Omega_2(\tau)\mathrm{Sp}F^2\right]g^{\mu\nu} + \Omega_2(\tau)(F^2)^{\mu\nu}, \tag{A.13}$$

where, due to the first relation in (4.5), either $(\tilde{F}^2)^{\mu\nu}$ or $(F^2)^{\mu\nu}$ can be chosen as complimentary to $g^{\mu\nu}$. Similarly to (A.3), the scalar functions in (A.13) are related to the appropriate traces:

$$\Omega_1(\tau) = \frac{1}{4}\mathrm{Sp}\lambda_5(\tau) + \frac{1}{4}\mathrm{Sp}F^2 \frac{4\mathrm{Sp}F^2\lambda_5(\tau) - \mathrm{Sp}F^2\mathrm{Sp}\lambda_5(\tau)}{(\mathrm{Sp}F^2)^2 + (\mathrm{Sp}F\tilde{F})^2},$$
$$\Omega_2(\tau) = \frac{4\mathrm{Sp}F^2\lambda_5(\tau) - \mathrm{Sp}F^2\mathrm{Sp}\lambda_5(\tau)}{(\mathrm{Sp}F^2)^2 + (\mathrm{Sp}F\tilde{F})^2}. \qquad (A.14)$$

Using (A.9), (A.10) and the first two relations in (4.5), we get

$$\Omega_1(\tau) = -8C_0^2(C_2^2 + C_4^2) = -\frac{(\mathrm{Sp}F\tilde{F})^2[(\mathrm{Sp}F\cot F\tau)^2 + (\mathrm{Sp}\tilde{F}\cot F\tau)^2]}{8[(\mathrm{Sp}F^2)^2 + (\mathrm{Sp}F\tilde{F})^2]}, \qquad (A.15)$$

and

$$\Omega_2(\tau) = 2[1 - 2C_3^{-2}C_4(C_2\mathrm{Sp}F\tilde{F} - C_4\mathrm{Sp}F^2)] = 2 + \mathrm{Sp}F\cot F\tau \frac{\mathrm{Sp}F^2\mathrm{Sp}F\cot F\tau + \mathrm{Sp}F\tilde{F}\mathrm{Sp}\tilde{F}\cot F\tau}{(\mathrm{Sp}F^2)^2 + (\mathrm{Sp}F\tilde{F})^2}. \qquad (A.16)$$

Using (A.9) and (4.16), we reduce (4.9) to the form

$$\rho^{\mu\nu}(\tau) = 2C_0[C_1 g^\mu_\beta - 2(C_2F^{\mu\alpha} + C_4\tilde{F}^{\mu\alpha}) \times (\cot F\tau)_{\alpha\beta}] \left(\tilde{F}\frac{\sin^2 F\tau}{F}\right)^{\beta\nu}, \qquad (A.17)$$

and, similarly, (4.11) with the use of (4.16) is reduced to the form

$$\rho_5^{\mu\nu}(\tau) = -C_0[C_1 g^\mu_\beta - 2(C_2F^{\mu\alpha} + C_4\tilde{F}^{\mu\alpha})(\cot F\tau)_{\alpha\beta}] \times \left[(\sin 2F\tau)^{\beta\nu} - \left(\frac{\sin^2 F\tau}{F}\right)^{\beta\nu} \mathrm{Sp}F\cot F\tau\right]. \qquad (A.18)$$

Thus, in order to prove the vanishing of $\rho^{\mu\nu}$ and $\rho_5^{\mu\nu}$, it is sufficient to prove the relation

$$C_1 g^\mu_\beta - 2(C_2F^{\mu\alpha} + C_4\tilde{F}^{\mu\alpha})(\cot F\tau)_{\alpha\beta} = 0. \qquad (A.19)$$

First, using again (4.16), we get

$$(C_2F^{\mu\alpha} + C_4\tilde{F}^{\mu\alpha})(\cot F\tau)_{\alpha\beta} = g^\mu_\beta(2C_3)^{-1}[(C_2^2 - C_4^2)\mathrm{Sp}F\tilde{F} - 2C_2C_4\mathrm{Sp}F^2]. \qquad (A.20)$$

Then, using the explicit form of C_j in (4.13), we find the relation

$$C_1C_3 = (C_2^2 - C_4^2)\mathrm{Sp}F\tilde{F} - 2C_2C_4\mathrm{Sp}F^2, \qquad (A.21)$$

which, together with the previous relation, proves (A.19) and, consequently, (4.15). Thus, $\rho^{\mu\nu}$ and $\rho_5^{\mu\nu}$ are equal to zero, and, in view of (A.11), $\omega^{\mu\nu}$ is also equal to zero, whereas $\omega_5^{\mu\nu}$ is equal to $\lambda_5^{\mu\nu}$ (A.13), which with the use of (4.5) is recast into the form with the dual field $\tilde{F}$ eliminated, resulting in (4.18).

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